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Sturm-Liouville Problem for Stationary Differential Operator with Nonlocal Two-Point Boundary Conditions

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Abstract. The Sturm-Liouville problem with various types of two-point boundary conditions is considered in this paper. In the first part of the paper, we investigate the Sturm-Liouville problem in three cases of nonlocal two-point boundary conditions. We prove general properties of the eigenfunctions and eigenvalues for such a problem in the complex case. In the second part, we investigate the case of real eigenvalues. It is analyzed how the spectrum of these problems depends on the boundary condition parameters. Qualitative behavior of all eigenvalues subject to the nonlocal boundary condition parameters is described.

Keywords: Sturm-Liouville problem, nonlocal two-point conditions.

1 Introduction

Boundary problems with nonlocal conditions are an area of the fast developing differential equations theory. Problems of this type arise in various fields of physics, biology, biotechnology, etc. Nonlocal conditions appear when the value of the function on the boundary is connected with the values inside the domain. Theoretical investigation of problems with various types of nonlocal boundary conditions is a topical problem and recently has been paid much attention to them in the scientific literature.

A. A. Samarskii and A.V. Bitsadze were originators of such problems. They formulated and investigated the nonlocal boundary problem for an elliptic equation [1]. J. Canon was one of the pioneers who investigated parabolic problems with integral boundary conditions [2]. Also parabolic problems with nonlocal integral boundary conditions were analyzed in [3–9].

A multipoint nonlocal boundary problem for second-order ordinary differential equations was initiated by V. Ilyin and E. Moiseev [10]. This problem was also investigated in [11–13]. During the last decades the number of differential problems with nonlocal boundary conditions increased significantly.

Quite new an area, related with the problems of this type, is investigation of a spectrum of differential equations with nonlocal conditions. Eigenvalue problems with nonlocal conditions are closely linked with boundary problems for differential equations with nonlocal conditions [14–16]. In [17–19], similar problems are investigated for the operators with a nonlocal condition of Bitsadze-Samarskii or integral type. Eigenvalue problems for differential operators with nonlocal conditions are considerably less investigated than the classical boundary condition cases.

The purpose of this paper is to analyze a real eigenvalue problem for a stationary differential problem with one classical and one nonlocal two-point boundary condition. In this paper, we analyze a stationary problem in three cases of such nonlocal boundary conditions. We investigate how the spectrum of these problems depends on the parameters of some nonlocal boundary conditions.

Some results on the spectrum of the problem with nonlocal Samarskii-Bitsadze type boundary condition are published in [20]. In [21], a similar Sturm-Liouville problem with two types of nonlocal integral boundary conditions was considered. The spectrum of those problems is very complicated for various cases of parameters – negative and complex eigenvalues may exist.

In Section 2, we analyze the Sturm-Liouville problem with a nonlocal two-point boundary condition and find general properties of eigenvalues and eigenfunctions in the complex plane. In Section 3, we investigate real eigenvalues in the case real parameters and we show that for some parameters one or two negative eigenvalues can exist. We investigate the cases when complex and multiple eigenvalues exist.

2 The Sturm-Liouville problem with a nonlocal two-point boundary condition

Let us analyze the Sturm-Liouville problem with one classical boundary condition

$$-u'' = \lambda u, \quad x \in (0,1), \tag{1}$$

$$u(0) = 0, \tag{2}$$

and another nonlocal two-point boundary condition of Samarskii-Bitsadze type:

$$\text{Case 1)} \qquad u'(1) = \gamma u(\xi), \tag{3_1}$$

$$\text{Case 2)} \qquad u'(1) = \gamma u'(\xi), \tag{3_2}$$

$$\text{Case 3)} \qquad u(1) = \gamma u'(\xi), \tag{3_3}$$

$$\text{Case 4)} \qquad u(1) = \gamma u(\xi), \tag{3_4}$$

with the parameters $\gamma \in \overline{\mathbb{C}} := \mathbb{C} \cup \infty$ and $\xi \in [0,1]$. The last case was analyzed in [20]. So, we investigate only the first cases (3_1),(3_2),(3_3). Note that the index in references denotes the case.

Remark 1. *For $\gamma = \infty$, we investigate boundary conditions with $\xi > 0$:*

$$u(\xi) = 0 \text{ (*Cases* 1,4)}, \quad u'(\xi) = 0 \text{ (*Cases* 2,3)}$$

instead of boundary conditions (3).

In this section, we will find all eigenvalues, which do not depend on the parameter γ, and we will show how they depend on the parameter ξ. We will formulate the basic properties for eigenvalues, which are depending on the parameter γ.

Let us define $\mathbb{N} := \{1, 2, \ldots\}$. Denote by $\mathbb{N}_k := \{j \in \mathbb{N} | j/k \in \mathbb{N}\}$, $k \in \mathbb{N}$, a subset of integer positive numbers, by $\mathbb{N}_e = \mathbb{N}_2 \cup \{0\}$ a set of even nonnegative integer numbers, and $\mathbb{N}_o = \mathbb{N} \setminus \mathbb{N}_2$ a set of odd positive integer numbers. Let $r = \frac{m}{n} \in \mathbb{Q}[0,1]$ be a rational number in $[0,1]$. For $r \in (0,1)$, we suppose that m and n $(n > m > 0)$ are positive coprime integer numbers. If $r = 0$, we suppose $m = 0$, $n = 1$ and, if $r = 1$, we suppose $m = 1$, $n = 1$.

When $\gamma = 0$ in the problem (1)–(3), we get a problem with classical boundary conditions. Then eigenvalues and eigenfunctions don't depend on the parameter ξ:

$$\lambda_k = \pi^2\bigl(k - \tfrac{1}{2}\bigr)^2, \quad u_k(x) = \sin\bigl(\pi(k - \tfrac{1}{2})x\bigr), \quad k \in \mathbb{N}, \tag{$4_{1,2}$}$$

$$\lambda_k = (\pi k)^2, \qquad u_k(x) = \sin(\pi k x), \qquad k \in \mathbb{N}. \tag{$4_{3,4}$}$$

Remark 2. *We have the classical eigenfunctions and eigenvalues* (4) *if* $\xi = 0$ *in Cases* 1,4*. For* $\xi = 1$ *in Cases* 2,4 *we have the classical eigenfunctions and eigenvalues* (4) *only for* $\gamma \neq 1$ *and the second boundary condition is trivial for* $\gamma = 1$*. For* $\xi = 1$ *in Cases* 1,3 *we have third type (classical) boundary condition.*

If $\lambda = 0$, then all the functions $u(x) = cx$ satisfy the problem (1)–(2). Substituting this solution into the second boundary condition (3), we get the equalities: $c = c\gamma\xi$ (Case $1,4$), $c = c\gamma$ (Case $2,3$).

Lemma 1. *The eigenvalue* $\lambda = 0$ *exists if and only if:* $\gamma = \frac{1}{\xi}$ *in Cases* 1,4*;* $\gamma = 1$ *in Cases* 2,3*.*

In the general case, for $\lambda \neq 0$, eigenfunctions are $u = c\sin(qx)$ and eigenvalues $\lambda = q^2$, where $q \in \mathbb{C}_q \setminus \{0\}$,

$$\mathbb{C}_q := \{q \in \mathbb{C}|\ \operatorname{Re} q > 0 \text{ or } \operatorname{Re} q = 0, \operatorname{Im} q > 0 \text{ or } \ q = 0\}.$$

These eigenfunctions satisfy equation (1), boundary condition (2) and nonlocal boundary condition (3). As $\lambda \neq 0$ the nonlocal boundary condition is satisfied if

$$cq\cos q = c\gamma\sin(\xi q), \tag{5_1}$$

$$cq\cos q = c\gamma q\cos(\xi q), \tag{5_2}$$

$$c\sin q = c\gamma q\cos(\xi q) \tag{5_3}$$

and there exists a nontrivial solution if $z = q$ is the root of the equation

$$f_1(z) := \gamma\frac{\sin(\xi z)}{z} - \cos z = 0, \tag{6_1}$$

$$f_2(z) := \gamma\cos(\xi z) - \cos z = 0, \tag{6_2}$$

$$f_3(z) := \gamma\cos(\xi z) - \frac{\sin z}{z} = 0. \tag{6_3}$$

If $\sin(\xi q) = 0$ and $\cos q = 0$ in Case 1, $\cos(\xi q) = 0$ and $\cos q = 0$ in Case 2 or $\cos(\xi q) = 0$ and $\sin q = 0$ in Case 3, then equation (6) is valid for all $\gamma \in \mathbb{C}$. In this case, we get *constant eigenvalues* $\lambda = q^2$, which don't depend on the parameter γ, and q is the root of the system:

$$\cos q = 0, \qquad \sin(\xi q) = 0, \tag{7_1}$$

$$\cos q = 0, \qquad \cos(\xi q) = 0, \tag{7_2}$$

$$\sin q = 0, \qquad \sin(\xi q) = 0. \tag{7_3}$$

If $\lambda = q^2$ is a constant eigenvalue, then we will name $q \in \mathbb{C}_q$ *constant eigenvalue point*.

Proposition 1. *If the parameter ξ is an irrational number, then constant eigenvalues do not exist.*

Proof. The roots of the first equation are $q_k = \pi(k - \frac{1}{2})$ (Cases 1, 2) or $q_k = \pi k$ (Case 3), $k \in \mathbb{N}$, the roots of the second equation are $q_l = \pi(l - \frac{1}{2})/\xi$ (Case 2) or $q_l = \pi l/\xi$ (Cases 1, 3), $l \in \mathbb{N}$. The numbers $q_k/\pi \in \mathbb{Q}$, but $q_l/\pi \notin \mathbb{Q}$. So, system (7) has no solutions. $\square$

Remark 3. *In Case* 4 *(see,* [20]*) constant eigenvalues exist only for rational* $\xi = r = \frac{m}{n} \in [0, 1)$ *and they are equal to* $\lambda_k = (\pi n k)^2$, $k \in \mathbb{N}$.

Proposition 2. *Let n and m ($0 < m < n$) be coprime numbers and $z \in \mathbb{C}_q \setminus \{0\}$. Then*

$$\begin{cases} \cos(nz) = 0, \\ \sin(mz) = 0 \end{cases} \sim \quad \begin{array}{l} \cos z = 0, \ \textit{for } m \in \mathbb{N}_e, n \in \mathbb{N}_o, \\ \varnothing \ \textit{otherwise}; \end{array} \tag{8_1}$$

$$\begin{cases} \cos(nz) = 0, \\ \cos(mz) = 0 \end{cases} \sim \quad \begin{array}{l} \cos z = 0, \ \textit{for } m \in \mathbb{N}_o, n \in \mathbb{N}_o, \\ \varnothing \ \textit{otherwise}; \end{array} \tag{8_2}$$

$$\begin{cases} \sin(nz) = 0, \\ \cos(mz) = 0 \end{cases} \sim \quad \begin{array}{l} \cos z = 0, \ \textit{for } m \in \mathbb{N}_o, n \in \mathbb{N}_e, \\ \varnothing \ \textit{otherwise}; \end{array} \tag{8_3}$$

$$\begin{cases} \sin(nz) = 0, \\ \sin(mz) = 0 \end{cases} \sim \quad \sin z = 0. \tag{8_4}$$

Proof. Case 1. Positive roots of equations $\cos(nz) = 0$ and $\sin(mz) = 0$ are $\frac{\pi}{2}\frac{2k-1}{n}$, $k \in \mathbb{N}$, and $\frac{\pi}{2}\frac{2l}{m}$, $l \in \mathbb{N}$, accordingly. We have the common root if $(2k-1)m = 2ln$. So, such roots exist only if m is even. If n and m are coprime numbers, then n must be odd and $2k-1 = n\cdot(2\tilde{k}-1)$, $\tilde{k} \in \mathbb{N}$, $2l = m\cdot\tilde{l}$, $\tilde{l} \in \mathbb{N}$. If $\tilde{l} = 2\tilde{k}-1$ then we get the common root, i.e., $z_k = \pi(\tilde{k}-\frac{1}{2})$, $\tilde{k} \in \mathbb{N}$. Those roots (and only they) are the roots of the equation $\cos z = 0$.

Case 2. Positive roots of equations $\cos(nz) = 0$ and $\cos(mz) = 0$ are $\frac{\pi}{2}\frac{2k-1}{n}$, $k \in \mathbb{N}$, and $\frac{\pi}{2}\frac{2l-1}{m}$, $l \in \mathbb{N}$, respectively. We have the common root if $(2k-1)m = (2l-1)n$. Thus, such roots exist if m and n both are the odd numbers. Then $2k-1 = n\cdot(2\tilde{k}-1)$, $\tilde{k} \in \mathbb{N}$, $2l-1 = m\cdot(2\tilde{l}-1)$, $\tilde{l} \in \mathbb{N}$. If $\tilde{l} = \tilde{k}$, then we get the common root $z_k = \pi(\tilde{k}-\frac{1}{2})$, $\tilde{k} \in \mathbb{N}$, i.e., root of the equation $\cos z = 0$.

Case 3. Positive roots of equations $\sin(nz) = 0$ and $\cos(mz) = 0$ are $\frac{\pi}{2}\frac{2k}{n}$, $k \in \mathbb{N}$, and $\frac{\pi}{2}\frac{2l-1}{m}$, $l \in \mathbb{N}$, respectively. We have the common root if $2km = (2l-1)n$. Thus, such roots exist if n is even. Then m must be odd and $2k = n\cdot\tilde{k}$, $\tilde{k} \in \mathbb{N}$, $2l-1 = m\cdot(2\tilde{l}-1)$, $\tilde{l} \in \mathbb{N}$. If $\tilde{k} = 2\tilde{l}-1$, then we get the common root $z_k = \pi(\tilde{k}-\frac{1}{2})$, $\tilde{k} \in \mathbb{N}$, i.e., the root of the equation $\cos z = 0$.

Case 4. Positive roots of equations $\sin(nz) = 0$ and $\sin(mz) = 0$ are $\pi k/n$, $k \in \mathbb{N}$, and $\pi l/m$, $l \in \mathbb{N}$, respectively. We have the common root if $km = ln$. Consequently, there exist such roots if $k = n\cdot\tilde{k}$, $\tilde{k} \in \mathbb{N}$, $l = m\cdot\tilde{l}$, $\tilde{l} \in \mathbb{N}$. If $\tilde{l} = \tilde{k}$, then we get the common root $z_k = \pi\tilde{k}$, $\tilde{k} \in \mathbb{N}$, i.e., the root of the equation $\sin z = 0$. □

Lemma 2. *Constant eigenvalues do not exist for irrational ξ, while for rational $\xi = r = \frac{m}{n} \in [0,1]$ they exist in the following cases:*

$m \in \mathbb{N}_e$, $n \in \mathbb{N}_o$ *in Case* 1;

$m \in \mathbb{N}_o$, $n \in \mathbb{N}_o$ $m \le n$ *in Case* 2;

$m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$ *in Case* 3;

and constant eigenvalues are equal to $\lambda_k = c_k^2$, $c_k = \pi(k-\frac{1}{2})n$, $k \in \mathbb{N}$.

Proof. If ξ is the irrational number then, this Lemma follows from Proposition 1. If $z = q/n$, $q \in \mathbb{C}_q$ in Proposition 2, prove all the statements of Lemma 2 in the case of rational $\xi = r \in (0,1)$.

If $\xi = 0$, then from equation (6) it follows that we have only constant eigenvalues in Case 1 (classical case), and there are no constant eigenvalues in Cases 2, 3. If $\xi = 1$ there are no constant eigenvalues in Cases 1, 3 (the third type boundary condition), because the functions $\sin q$ and $\cos q$ have no common zeroes, and we get only constant eigenvalues in Case 2 (classical case $\gamma \neq 1$). $\square$

Let us define the functions $S_j(z) := \frac{\sin(jz)}{\cos z}$, $j \in \mathbb{N}_e$, $C_j(z) := \frac{\cos(jz)}{\cos z}$, $j \in \mathbb{N}_o$. We can express them by the Moivre formula:

$$
\begin{aligned}
S_{2k}(z) &= 2k\cos^{2k-2} z \sin z - \tbinom{2k}{3}\cos^{2k-4} z \sin^3 z + \dots \\
&\qquad\qquad + (-1)^{k-1} 2k \sin^{2k-1} z, \\
C_{2k+1}(z) &= \cos^{2k} z - \tbinom{2k+1}{2}\cos^{2k-2} z \sin^2 z + \dots \\
&\qquad\qquad + (-1)^{k}\sin^{2k} z, \quad k \in \mathbb{N}\cup\{0\}.
\end{aligned}
$$

We can see that $S_0(z) \equiv 0$, $C_1(z) \equiv 1$ and, for $k \geq 1$, the functions $S_{2k}(z)$ and $C_{2k+1}(z)$ are entire transcendental with the first order of growth. The functions $\cos z$ and $S_j(z)$, $j > 1$, and $C_j(z)$ don't have common zeroes (see, Proposition 2).

Remark 4. *The functions $S_{2k}(z) = \sin z P_{2k}(\cos z)$, $C_{2k+1}(z) = P_{2k+1}(\cos z)$, where P_j, $j \in \mathbb{N}\cup\{0\}$ are polynomials (with real integer coefficients):*

$$
\begin{aligned}
P_{2k}(z) &= 2kz^{2k-2} - \tbinom{2k}{3} z^{2k-4}(1-z^2) + \dots + (-1)^{k-1}2k(1-z^2)^{k-1}, \\
P_{2k+1}(z) &= z^{2k} - \tbinom{2k+1}{2} z^{2k-2}(1-z^2) + \dots + (-1)^{k}(1-z^2)^{k},
\end{aligned}
$$

and $P_0 \equiv 0$, $P_1 \equiv 1$, $P_2 \equiv 1$, and P_k are nonconstant for $k > 2$.

Let us denote the sets:

$$
\begin{aligned}
&\Xi := \big\{\xi \in [0,1] \,|\, \text{for } \xi \text{ constant eigenvalues don't exist}\big\}; \\
&R := \big\{\xi \in [0,1] \,|\, \text{for } \xi \text{ there exist constant and nonconstant eigenvalues}\big\}; \\
&C := \big\{\xi \in [0,1] \,|\, \text{for } \xi \text{ there exist only constant eigenvalues}\big\}.
\end{aligned}
$$

>From Lemma 2 we have:

$$
\begin{aligned}
R &= \Big\{\xi = \frac{m}{n} \,\Big|\, m \in \mathbb{N}_e, n \in \mathbb{N}_o, 0 < m < n\Big\} && \text{in Case 1;} \\
R &= \Big\{\xi = \frac{m}{n} \,\Big|\, m \in \mathbb{N}_o, n \in \mathbb{N}_o, m < n\Big\} && \text{in Case 2;} \\
R &= \Big\{\xi = \frac{m}{n} \,\Big|\, m \in \mathbb{N}_o, n \in \mathbb{N}_e, m < n\Big\} && \text{in Case 3;}
\end{aligned}
$$

and

$$\begin{aligned}
&\Xi = \big((0,1) \setminus R\big) \cup \{1\}, && C = \{0\} && \text{in Case 1;}\\
&\Xi = \big((0,1) \setminus R\big) \cup \{0\}, && C = \{1\} && \text{in Case 2;}\\
&\Xi = \big((0,1) \setminus R\big) \cup \{0,1\}, && C = \varnothing && \text{in Case 3.}
\end{aligned}$$

Remark 5. *In Case* 2 *and* $\xi = 1$*, for* $\gamma \neq 1$ *we have constant eigenvalues, but for* $\gamma = 1$ *we don't have the second boundary condition (it is trivial). In this special case, we define* $\xi = 1 \in C$*. Then* $R \cup \Xi \cup C = [0,1]$ *in the all cases.*

For $\xi \in R$ from equation (6) we get:

$$\cos\frac{q}{n} \cdot \left[\frac{\gamma}{q} S_m\Big(\frac{q}{n}\Big) - C_n\Big(\frac{q}{n}\Big)\right] = 0, \tag{9_1}$$

$$\cos\frac{q}{n} \cdot \left[\gamma C_m\Big(\frac{q}{n}\Big) - C_n\Big(\frac{q}{n}\Big)\right] = 0, \tag{9_2}$$

$$\cos\frac{q}{n} \cdot \left[\gamma C_m\Big(\frac{q}{n}\Big) - \frac{1}{q} S_n\Big(\frac{q}{n}\Big)\right] = 0. \tag{9_3}$$

Let us analyze nonconstant eigenvalues. For $\xi \in R$, let us define the functions:

$$f_{1r}(z) := \gamma\frac{\sin\frac{z}{n}}{z} P_m\Big(\cos\frac{z}{n}\Big) - P_n\Big(\cos\frac{z}{n}\Big), \tag{10_1}$$

$$f_{2r}(z) := \gamma P_m\Big(\cos\frac{z}{n}\Big) - P_n\Big(\cos\frac{z}{n}\Big), \tag{10_2}$$

$$f_{3r}(z) := \gamma P_m\Big(\cos\frac{z}{n}\Big) - \frac{\sin\frac{z}{n}}{z} P_n\Big(\cos\frac{z}{n}\Big). \tag{10_3}$$

Remark 6. *In the case* $\gamma = \infty$*, we define:*

$$\begin{aligned}
&f_1(z) := \frac{\sin(\xi z)}{z}, \quad f_2(z) := \cos(\xi z), \quad f_3(z) := \cos(\xi z);\\
&f_{1r}(z) := \frac{\sin\frac{z}{n}}{z} P_m\Big(\cos\frac{z}{n}\Big), \quad f_{2r}(z) := P_m\Big(\cos\frac{z}{n}\Big),\\
&f_{3r}(z) := P_m\Big(\cos\frac{z}{n}\Big).
\end{aligned}$$

Note that $f_{2r} \equiv 1$ *for* $\xi = \frac{1}{3}$ *and* $f_{3r} \equiv 1$ *for* $\xi = \frac{1}{2}$*.*

Lemma 3. *There is a countable number of nonconstant eigenvalues for every* $\gamma \in \mathbb{C}$ *and every* $\xi \in \Xi \cup R$*. The point* $\lambda = \infty$ *is an accumulation point of those eigenvalues.*

Proof. The functions $f_k(\sqrt{\lambda})$, $k = 1,2,3$ for $\xi \in \Xi$ and functions $f_{kr}(\sqrt{\lambda})$, $k = 1,2,3$ for $\xi \in R$ are entire transcendental functions with the order of growth equal to $\frac{1}{2}$. Such functions acquire every γ-value for infinite (countable) times, and $\lambda = \infty$ is the accumulation point of γ-values (see, [22]). $\square$

We can get all nonconstant eigenvalues (which depend on the parameter γ) as γ-values of meromorphic functions defined on the set $\mathbb{C}_q$:

$$\gamma_1(z) := \frac{z\cos z}{\sin(\xi z)}, \tag{11_1}$$

$$\gamma_2(z) := \frac{\cos z}{\cos(\xi z)}, \tag{11_2}$$

$$\gamma_3(z) := \frac{\sin z}{z\cos(\xi z)} \tag{11_3}$$

when $\xi \in \Xi$ and

$$\gamma_{1r}(z) := n\frac{\frac{z}{n}}{\sin\frac{z}{n}}\frac{P_n\left(\cos\frac{z}{n}\right)}{P_m\left(\cos\frac{z}{n}\right)}, \tag{12_1}$$

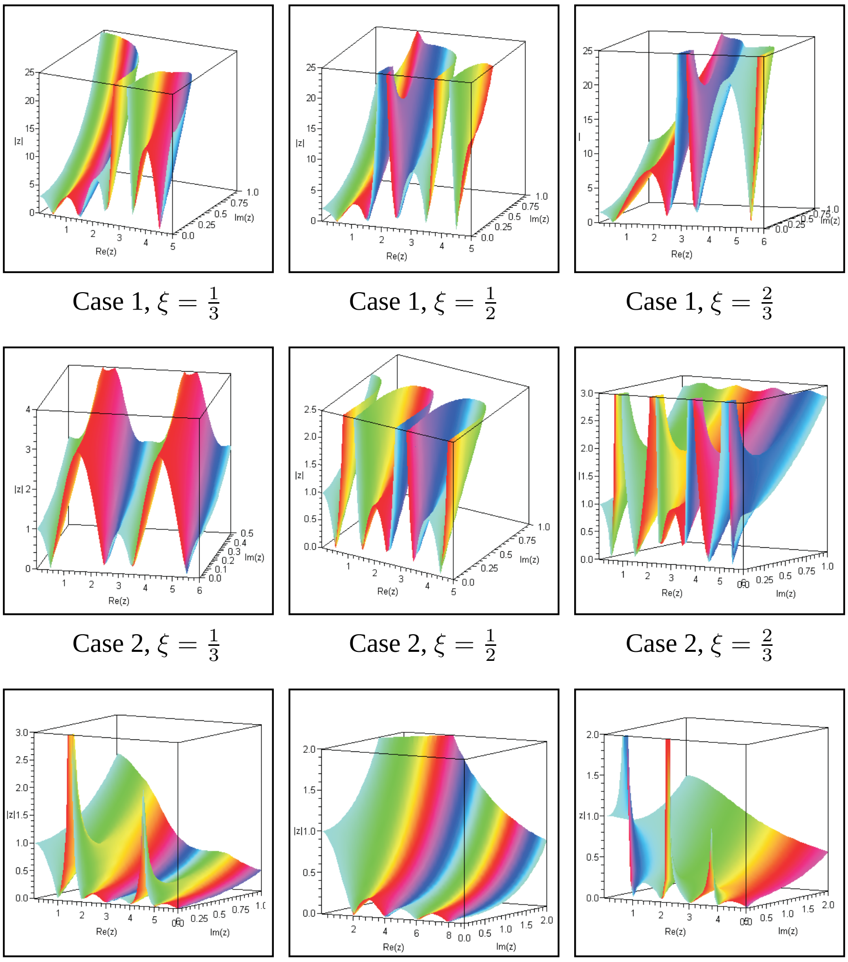
$$\gamma_{2r}(z) := \frac{P_n\left(\cos\frac{z}{n}\right)}{P_m\left(\cos\frac{z}{n}\right)}, \tag{12_2}$$

$$\gamma_{3r}(z) := \frac{1}{n}\frac{\sin\frac{z}{n}}{\frac{z}{n}}\frac{P_n\left(\cos\frac{z}{n}\right)}{P_m\left(\cos\frac{z}{n}\right)} \tag{12_3}$$

when $\xi = \frac{m}{n} \in R$.

Remark 7. *The poles of the function* $\gamma_k(q)$, $\gamma_{kr}(q)$, $k = 1,2,3$ *are eigenvalues of the problem* (1)–(3) *in the case* $\gamma = \infty$.

The graphs of functions $|\gamma_{kr}(z/\pi)|$, $k = 1,2,3$ for various rational ξ are presented in Fig. 1. As $\overline{\cos z} = \cos\bar{z}$, $\overline{\sin z} = \sin\bar{z}$, and polynomials P_n have real coefficients, we get a similar property for the functions γ_k and γ_{kr}: $\overline{\gamma(z)} = \gamma(\bar{z})$, $\overline{\gamma_r(z)} = \gamma_r(\bar{z})$. So, graphs are drawn only for $\operatorname{Im} z \geq 0$ and $\operatorname{Re} z \geq 0$. In the graphs the function $\gamma_{kr}(z/\pi)$ is drawn instead the function $\gamma_{kr}(z)$. In this case, the zeroes of the first function are the points k, $k \in \mathbb{N}$. A module of the complex functions display zeroes and poles of this function. In all showed graphics, the zeroes and the poles of the function are in the real axis, and function increases itself, when $\operatorname{Im} z$ grows up. Now we formulate main properties of these functions as a proposition and remarks.



Case 1, $\xi = \frac{1}{3}$ Case 1, $\xi = \frac{1}{2}$ Case 1, $\xi = \frac{2}{3}$

Case 2, $\xi = \frac{1}{3}$ Case 2, $\xi = \frac{1}{2}$ Case 2, $\xi = \frac{2}{3}$

Case 3, $\xi = \frac{1}{3}$ Case 3, $\xi = \frac{1}{2}$ Case 3, $\xi = \frac{2}{3}$

Fig. 1. Functions $|\gamma_{kr}(z/\pi)|$ for various ξ.

Proposition 3. *All zeroes and poles of the meromorphic functions γ_k, γ_{kr}, lie on the positive part of the real axis.*

Proof. The proof follows directly from (11) and (12) and the properties of sinus and cosinus functions (all zeroes of these functions are real numbers). So, on $\mathbb{C}_q$ we have only positive zeroes and poles. $\square$

We note that for $\xi = \frac{1}{3}$ in Case 2 and for $\xi = \frac{1}{2}$ in Case 3 there are no poles.

Remark 8. *If $\xi \in \Xi$, then $z_j = \pi(k - \frac{1}{2})$, $k \in \mathbb{N}$ are zeroes (the first order) of the functions $\gamma_1(z)$, $\gamma_2(z)$, and $z_k = \pi k$, $k \in \mathbb{N}$ are zeroes (the first order) of the function $\gamma_3(z)$ (see, formula (11)). The points $p_k = \pi k/\xi$, $k \in \mathbb{N}$ are poles (the first order) of the function $\gamma_1(z)$, and points $p_k = \pi(k - \frac{1}{2})/\xi$, $k \in \mathbb{N}$ are poles (the first order) of the functions $\gamma_2(z)$, $\gamma_3(z)$.*

Remark 9. *If $\xi \in R$, at the points of constant eigenvalues $c_k = \pi(k - \frac{1}{2})n$, $k \in \mathbb{N}$ (see Lemma 2 and Remark 8) zeroes correspond to the poles. Thus, the functions γ_{kr} are analytic at those points:*

$$\begin{aligned}\lim_{q\to c_k} \gamma_1 = \gamma_{1r}(c_k;\xi) &= (-1)^{(n-m-1)/2}(-1)^k(k - \tfrac{1}{2})\frac{n^2}{m}\pi \\ &= (-1)^{(n-m-1)/2}(-1)^k\frac{c_k}{\xi}, \qquad (13_1)\end{aligned}$$

$$\lim_{q\to c_k} \gamma_2 = \gamma_{2r}(c_k;\xi) = (-1)^{(n-m)/2}\frac{n}{m} = (-1)^{(n-m)/2}\frac{1}{\xi}, \qquad (13_2)$$

$$\begin{aligned}\lim_{q\to c_k} \gamma_3 = \gamma_{3r}(c_k;\xi) &= (-1)^{(n-m-1)/2}(-1)^k\frac{1}{m(k - \frac{1}{2})\pi} \\ &= (-1)^{(n-m-1)/2}(-1)^k\frac{1}{c_k\xi} \qquad (13_3)\end{aligned}$$

and $\gamma_l(c_k) \neq 0$, $l = 1, 2, 3$. We use the notation $f(x;\xi)$ or $f(x)$ when investigating the function $f(x,\xi)$ as a one-dimensional one with the fixed parameter ξ or parameters $\xi = (\xi_1, \xi_2)$.

Let $c \in (a, b)$ and f, g be real functions in $C^2(a, b)$ with the properties $f'' = \alpha f$, $g'' = \beta g$, $f(c) = g(c) = 0$, $g \neq 0$ for $q \neq c$, $g'(c) \neq 0$ and $\tilde{\gamma}(q) = \frac{f(q)}{g(q)}$, $\lim_{q\to c}\tilde{\gamma}(q) = \lim_{q\to c}\frac{f(g)}{g(q)} = \tilde{\gamma}_c$. Then

$$\begin{aligned}\lim_{q\to c}\frac{\tilde{\gamma}'g'}{g} &= g'(c)\lim_{q\to c}\frac{f'g - fg'}{g^3} = g'(c)\lim_{q\to c}\frac{f''g - fg''}{3g^2g'} \\ &= \lim_{q\to c}\frac{\alpha fg - \beta fg}{3g^2} = \frac{\alpha - \beta}{3}\lim_{q\to c}\frac{f}{g} = \frac{\alpha - \beta}{3}\tilde{\gamma}_c, \\ \lim_{q\to c}\tilde{\gamma}'' &= \lim_{q\to c}\Big(\frac{f'g - fg'}{g^2}\Big)' = \lim_{q\to c}\frac{f''g - fg''}{g^2} - \lim_{q\to c}(f'g - fg')\frac{2g'}{g^3} \\ &= \lim_{q\to c}\frac{(\alpha - \beta)fg}{g^2} - 2\lim_{q\to c}\tilde{\gamma}'\frac{g'}{g} \\ &= (\alpha - \beta)\tilde{\gamma}_c - \frac{2}{3}(\alpha - \beta)\tilde{\gamma}_c = \frac{\alpha - \beta}{3}\tilde{\gamma}_c.\end{aligned}$$

If $|\tilde\gamma_c| < \infty$*, then* $\lim_{q\to c}\tilde\gamma' = \lim_{q\to c}\dfrac{\tilde\gamma' g'}{g}\lim_{q\to c}\dfrac{g}{g'} = \dfrac{\alpha-\beta}{3}\tilde\gamma_c\cdot\dfrac{0}{g'(c)} = 0$*. In particular, we have*

$$\lim_{q\to c_k}\Big(\frac{\cos q}{\sin(\xi q)}\Big)' = 0, \quad \lim_{q\to c_k}\Big(\frac{\cos q}{\sin(\xi q)}\Big)'' = -\frac{1-\xi^2}{3}\frac{\cos c_k}{\sin(\xi c_k)}, \tag{14_1}$$

$$\lim_{q\to c_k}\Big(\frac{\cos q}{\cos(\xi q)}\Big)' = 0, \quad \lim_{q\to c_k}\Big(\frac{\cos q}{\cos(\xi q)}\Big)'' = -\frac{1-\xi^2}{3}\frac{\cos c_k}{\cos(\xi c_k)}, \tag{14_2}$$

$$\lim_{q\to c_k}\Big(\frac{\sin q}{\cos(\xi q)}\Big)' = 0, \quad \lim_{q\to c_k}\Big(\frac{\sin q}{\cos(\xi q)}\Big)'' = -\frac{1-\xi^2}{3}\frac{\sin c_k}{\cos(\xi c_k)}, \tag{14_3}$$

and

$$\gamma_1'(c_k;\xi) = \lim_{q\to c_k}\Big(\frac{q\cos q}{\sin(\xi q)}\Big)' = \frac{\gamma_{1+}(c_k;\xi)}{c_k}, \tag{15_1}$$

$$\gamma_2'(c_k;\xi) = \lim_{q\to c_k}\Big(\frac{\cos q}{\cos(\xi q)}\Big)' = 0, \tag{15_2}$$

$$\gamma_3'(c_k;\xi) = \lim_{q\to c_k}\Big(\frac{\sin q}{q\cos(\xi q)}\Big)' = -\frac{\gamma_{3+}(c_k;\xi)}{c_k}, \tag{15_3}$$

$$\begin{aligned}\gamma_1''(c_k;\xi) &= \lim_{q\to c_k}\Big(\frac{q\cos q}{\sin(\xi q)}\Big)'' = 2\lim_{q\to c_k}\Big(\frac{\cos q}{\sin(\xi q)}\Big)' + \lim_{q\to c_k} q\Big(\frac{\cos(q)}{\sin(\xi q)}\Big)'' \\ &= 0 - c_k\frac{1-\xi^2}{3}\frac{\sin c_k}{\cos(\xi c_k)} = -\frac{1-\xi^2}{3}\gamma_1(c_k;\xi),\end{aligned} \tag{16_1}$$

$$\gamma_2''(c_k;\xi) = \lim_{q\to c_k}\Big(\frac{\cos q}{\cos(\xi q)}\Big)'' = -\frac{1-\xi^2}{3}\gamma_2(c_k;\xi), \tag{16_2}$$

$$\gamma_3''(c_k;\xi) = \lim_{q\to c_k}\Big(\frac{\sin q}{q\cos(\xi q)}\Big)'' = -\frac{1-\xi^2}{3}\gamma_3(c_k;\xi). \tag{16_3}$$

Remark 10. *We can enumerate all the poles* p_k*,* $k\in\mathbb{N}$*, in the increasing order* $p_1 < p_2 < \cdots < p_k < \ldots$*. Formally we denote* $p_0 = 0$*,* $p_\infty = +\infty$*. In the case* $\xi\in R$*, there can be only one term* p_0 *in the sequence* $\{p_k\}_{k=0}^{\infty}$*.*

Remark 11. *For* $\xi\in\mathbb{Q}$*, the functions* γ_l *and* γ_{lr} *are periodical or quasi-periodical in the real direction, i.e., if*

$$\tilde\gamma_1(z) := \frac{\gamma_1(z)}{z}, \qquad \tilde\gamma_{1r}(z) := \frac{\gamma_{1r}(z)}{z}, \tag{17_1}$$

$$\tilde\gamma_2(z) := \gamma_2(z), \qquad \tilde\gamma_{2r}(z) := \gamma_{2r}(z), \tag{17_2}$$

$$\tilde\gamma_3(z) := \gamma_3(z)z, \qquad \tilde\gamma_{3r}(z) := \gamma_{3r}(z)z, \tag{17_3}$$

then

$$\tilde{\gamma}_1(z+2\pi n)=\tilde{\gamma}_1(z), \qquad \tilde{\gamma}_{1r}(z+2\pi n)=\tilde{\gamma}_{1r}(z), \tag{18_1}$$

$$\tilde{\gamma}_2(z+2\pi n)=\tilde{\gamma}_2(z), \qquad \tilde{\gamma}_{2r}(z+2\pi n)=\tilde{\gamma}_{2r}(z), \tag{18_2}$$

$$\tilde{\gamma}_3(z+2\pi n)=\tilde{\gamma}_3(z), \qquad \tilde{\gamma}_{3r}(z+2\pi n)=\tilde{\gamma}_{3r}(z). \tag{18_3}$$

>From the inequalities $\sinh|\mathrm{Im}\,q| \le |\sin q|, |\cos q| \le \cosh(\mathrm{Im}\,q)$ we get the estimates

$$\frac{|z|\sinh|\mathrm{Im}\,z|}{\cosh(\xi\mathrm{Im}\,z)} \le |\gamma_1(z)|, \ \ |\gamma_{1r}(z)| \le \frac{|z|\cosh(\mathrm{Im}\,z)}{\sinh|\xi\mathrm{Im}\,z|}, \tag{19_1}$$

$$\frac{\sinh|\mathrm{Im}\,z|}{\cosh(\xi\mathrm{Im}\,z)} \le |\gamma_2(z)|, \ \ |\gamma_{2r}(z)| \le \frac{\cosh(\mathrm{Im}\,z)}{\sinh|\xi\mathrm{Im}\,z|}, \tag{19_2}$$

$$\frac{\sinh|\mathrm{Im}\,z|}{|z|\cosh(\xi\mathrm{Im}\,z)} \le |\gamma_3(z)|, \ \ |\gamma_{3r}(z)| \le \frac{\cosh(\mathrm{Im}\,z)}{|z|\sinh|\xi\mathrm{Im}\,z|}. \tag{19_3}$$

Corollary 1. *The next limits are valid:* $\lim\limits_{\mathrm{Im}\,q\to\pm\infty}\gamma_k=\infty$, $\lim\limits_{\mathrm{Im}\,q\to\pm\infty}\gamma_{kr}=\infty$, $k=1,2,3$, *except Cases* 2, 3 *for* $\xi=1$.

For the meromorphic function $F(z)$, we can define *a sign of a pole* at the point $z=p$:

$$\sigma_s(F,p)=\mathrm{sign}\big(\lim_{z\to p}(z-p)^sF(z)\big), s=0,1,\dots. \tag{20}$$

Remark 12. *If $\sigma_s(F,p)=0$, then the point $z=p$ is a pole and its order is lower than s or $z=p$ is analytical point; if $\sigma_s(F,p)=\infty$, then the point $z=p$ is a pole and its order is greater than s; otherwise we have an s-order pole.*

Remark 13. *If $F(z)=\frac{f(z)}{g(z)}$, where f,g are entire functions and $g(p)=0$, $g'(p)\neq 0$, then*

$$\sigma_1(F,p)=\mathrm{sign}\big(\lim_{z\to p}(z-p)F(z)\big)=\mathrm{sign}\,\mathrm{Res}_{z=p}\,F(z)=\mathrm{sign}\frac{f(p)}{g'(p)}. \tag{21}$$

3 Real eigenvalues case for the problem with two points nonlocal boundary condition

Let us consider the case where the parameter $\gamma\in\mathbb{R}$. Next we investigate the Sturm-Liouville problem (1)–(3) with real eigenfunctions and real eigenvalues $\lambda\in\mathbb{R}$.

Now, instead of $q \in \mathbb{C}_q$, we take q only in the rays $q = x \geq 0$ and $q = -\mathrm{i}x$, $x \leq 0$. We get positive eigenvalues in case the ray $q = x > 0$, and we have negative eigenvalues in the ray $q = -\mathrm{i}x$, $x < 0$. The point $q = x = 0$ corresponds to $\lambda = 0$. For the function $f : \mathbb{C}_q \to \mathbb{C}$, we have its two restrictions on those rays: $f_+(x) = f(x + \mathrm{i}0)$ for $x \geq 0$ and $f_-(x) = f(0 - \mathrm{i}x)$ for $x \leq 0$. The function f_+ corresponds to the case of positive eigenvalues, while the function f_- corresponds to that of negative eigenvalues. All real eigenvalues

$$\lambda_k = \begin{cases} x_k^2, & \text{for } x_k \geq 0, \\ -x_k^2, & \text{for } x_k \leq 0, \end{cases} \qquad k \in \mathbb{N},$$

are investigate of using the function $f : \mathbb{R} \to \mathbb{C}$:

$$f(x) = \begin{cases} f_+(x), & \text{for } x \geq 0, \\ f_-(x), & \text{for } x \leq 0. \end{cases}$$

For the complex functions (11), there such functions are real and can be written as:

$$\gamma_1(x;\xi) := \begin{cases} \gamma_{1-}(x;\xi) = \dfrac{x \cosh x}{\sinh(\xi x)}, & \text{for } x \leq 0, \\ \gamma_{1+}(x;\xi) = \dfrac{x \cos x}{\sin(\xi x)}, & \text{for } x \geq 0; \end{cases} \tag{22_1}$$

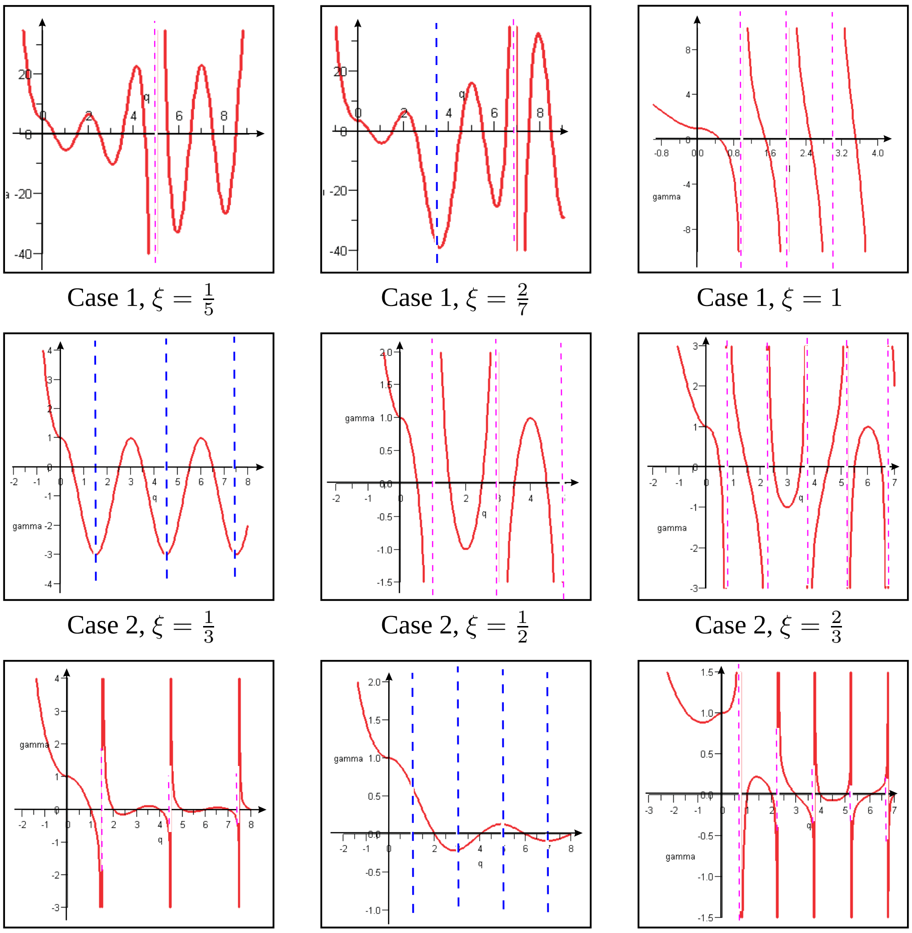
$$\gamma_2(x;\xi) := \begin{cases} \gamma_{2-}(x;\xi) = \dfrac{\cosh x}{\cosh(\xi x)}, & \text{for } x \leq 0, \\ \gamma_{2+}(x;\xi) = \dfrac{\cos x}{\cos(\xi x)}, & \text{for } x \geq 0; \end{cases} \tag{22_2}$$

$$\gamma_3(x;\xi) := \begin{cases} \gamma_{3-}(x;\xi) = \dfrac{\sinh x}{x \cosh(\xi x)}, & \text{for } x \leq 0, \\ \gamma_{3+}(x;\xi) = \dfrac{\sin x}{x \cos(\xi x)}, & \text{for } x \geq 0. \end{cases} \tag{22_3}$$

Graphs of the functions $\gamma_l(x;\xi)$, $l = 1,2,3$ for various ξ are shown in Fig. 2. Let us enumerate all the poles p_k, $k \in \mathbb{N}$ in the increasing order (see, Remark 10 in Section 2). The functions $\gamma_{1+}(x)$, $\gamma_{2+}(x)$ and $\gamma_{3+}(x)$ are defined in the intervals (p_{k-1}, p_k), $k \in \mathbb{N}$, where $p_{k-1} < p_k$ and $p_0 = 0$. All the functions $\gamma_{l-}(x) > 0$.

In the real case for $F = \gamma_{l+}$, $l = 1,2,3$, the sign of the poles $\sigma_1(F,p) = \pm 1$. If $\sigma_1(F,p) = 1$ (see Fig. 3(a)), then, for $\gamma >> 1$ the real eigenvalue point $q(\gamma) > p$

exists and $\lim_{\gamma\to+\infty} q(\gamma) = p$, for $\gamma << -1$ the real eigenvalue point $q(\gamma) < p$ exists and $\lim_{\gamma\to-\infty} q(\gamma) = p$, but there are no such points on the other side of the point p. If $\sigma_1(F, p) = -1$ (see Fig. 3(b)), then, for $\gamma >> 1$, the real eigenvalue point $q(\gamma) < p$ exists and $\lim_{\gamma\to+\infty} q(\gamma) = p$, as well as for $\gamma << -1$ the real eigenvalue point $q(\gamma) > p$ exists and $\lim_{\gamma\to-\infty} q(\gamma) = p$, but there are no such points on the other side of the point p. If $\sigma_1(F, p) = 0$ (see Fig. 3(c)), then for all γ there exists constant eigenvalue point $c = p$.

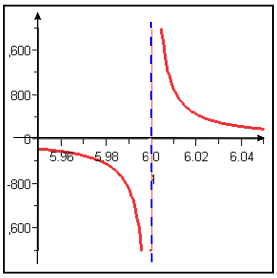


Case 1, $\xi = \frac{1}{5}$ | Case 1, $\xi = \frac{2}{7}$ | Case 1, $\xi = 1$

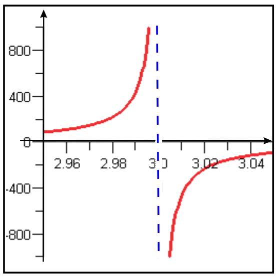
Case 2, $\xi = \frac{1}{3}$ | Case 2, $\xi = \frac{1}{2}$ | Case 2, $\xi = \frac{2}{3}$

Case 3, $\xi = \frac{1}{3}$ | Case 3, $\xi = \frac{1}{2}$ | Case 3, $\xi = \frac{2}{3}$

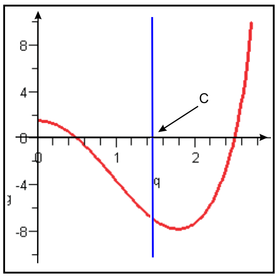
Fig. 2. Functions $\gamma_l(x/\pi)$, $l = 1, 2, 3$.



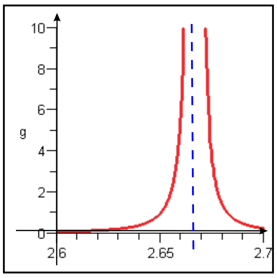
(a) $\sigma_1(F,p) = 1$



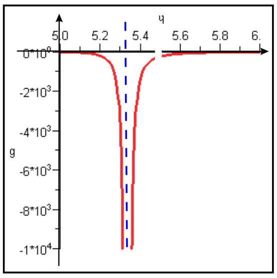
(b) $\sigma_1(F,p) = -1$



(c) $\sigma_1(F,c) = 0$



(d) $\sigma_2(F,p) = 1$



(e) $\sigma_2(F,p) = -1$

Fig. 3. The poles and constant eigenvalue points.

Remark 14. *Let the function F have the second order pole p. If $\sigma_2(F,p)=1$ (see Fig.* 3(d)*), then, for $\gamma >> 1$, two real eigenvalue points $q_1(\gamma) < p < q_2(\gamma)$ exist and $\lim_{\gamma\to+\infty} q_1(\gamma) = \lim_{\gamma\to+\infty} q_2(\gamma) = p$, but there are no such points for $\gamma << -1$. If $\sigma_2(F,p) = -1$ (see Fig.* 3(e)*), then, for $\gamma << -1$, two real eigenvalue points $q_1(\gamma) < p < q_2(\gamma)$ exist, while $\lim_{\gamma\to+\infty} q_1(\gamma) = \lim_{\gamma\to+\infty} q_2(\gamma) = p$ and there are no such points $\gamma >> 1$.*

Let us consider the equations:

$$\cos z - \gamma\sin(\xi z) = 0, \tag{23_1}$$

$$\cos z - \gamma\cos(\xi z) = 0, \tag{23_2}$$

$$\sin z - \gamma\cos(\xi z) = 0, \tag{23_3}$$

$$\sin z - \gamma\sin(\xi z) = 0. \tag{23_4}$$

We can prove (see, [20]) the next lemma, which is very useful for investigating real eigenvalues.

Lemma 4. *For real $\gamma \in [-1,1]$ and $\xi \in (0,1)$ all the roots of equations* (23) *are real numbers.*

3.1 Real eigenvalues in Case 1

Proposition 4. *The function $\gamma_{1-}(x;\xi)$ is a monotone decreasing function for $x < 0$ and all $\xi \in (0,1]$. The function $\gamma_{1+}(x;1)$ is a monotone decreasing function in each interval (p_{k-1}, p_k).*

Proof. The function $\gamma_{1-}(x)$ is even, when $x \in \mathbb{R}$, and $\gamma_{1-}(0) = \frac{1}{\xi}$ and $\gamma_{1-}(+\infty) = +\infty$. Therefore, we have to show that this function is increasing in interval $(0, +\infty)$.

Let us consider the function $y_1(x) := x \coth x$, $x > 0$. It is evident that $\sinh x > x$. So,

$$y_1'(x) = \frac{\sinh(2x) - 2x}{2\sinh^2 x} > 0,$$

and $y_1(x)$ is an increasing positive function for $x > 0$. Then $1/y_1(x) = \frac{1}{x}\tanh x$ is a decreasing positive function and its derivative is negative.

Let us consider the function $y(\xi, x) := \frac{1}{\xi}\tanh(\xi x) - \tanh x$, $x > 0$ and $\xi \in (0,1)$. For this function

$$\lim_{\xi \to 0+} y(\xi; x) = x - \tanh x > 0, \quad \lim_{\xi \to 1-} y(\xi; x) = 0 \tag{24}$$

for all $x > 0$. Its derivative with respect to ξ

$$y'(\xi; x) = \Big(\frac{1}{\xi}\tanh(\xi x)\Big)' = x\Big(\frac{1}{\xi x}\tanh(\xi x)\Big)' < 0.$$

So, $y(\xi; x)$ is a monotone decreasing function when $\xi \in (0,1)$, and from (24) we have that $y(\xi; x) > 0$ for all $\xi \in (0,1)$ and all $x > 0$.

Let us consider the function

$$y_2(x, \xi) := \frac{\sinh x}{\sinh(\xi x)}, \quad x > 0. \tag{25}$$

Its derivative with respect to x

$$\begin{aligned} y_2'(x; \xi) &= \frac{\cosh x \sinh(\xi x) - \xi \cosh(\xi x) \sinh x}{\sinh^2(\xi x)} \\ &= \frac{\xi y(\xi, x) \cosh x \cosh(\xi x)}{\sinh^2(\xi x)} > 0. \end{aligned}$$

Thus, $y_2(x;\xi)$ is an increasing positive function for all $x > 0$ and $\xi \in (0,1)$.

The function

$$\gamma_{1-}(x;\xi) = \frac{x\cosh x}{\sinh x}\frac{\sinh x}{\sinh(\xi x)} = y_1(x)\cdot y_2(x;\xi)$$

is a monotone increasing function for $x > 0$ as a product of monotone increasing positive functions. For $\xi = 1$, the function $y_2 \equiv 1$, and the proposition is valid in this case too.

Let us consider the function $\gamma_{1+}(x;1) = x\,\cot x$, $x > 0$, $x \neq k\pi$, $k \in \mathbb{N}$. It is evident that $\sin x < x$. So,

$$\gamma'_{1+}(x;1) = \frac{\sin(2x) - 2x}{2\sin^2 x} < 0,$$

and $\gamma_{1+}(x;1)$ is a monotone decreasing function in the intervals $(\pi(k-1),\pi k)$, $k \in \mathbb{N}$. □

In Section 2 we show that $\lambda = 0$ exists if and only if $\gamma = \gamma_0 = \frac{1}{\xi}$ (see, Lemma 1). Now from Proposition 2 we derive a few results for eigenvalues.

Lemma 5. *For $\gamma > \gamma_0$, there exists one negative eigenvalue, and for $\gamma \leq \gamma_0$, there are no negative eigenvalues.*

Proof. The function $\gamma_{1-}(x)$ is a monotone decreasing function when $x < 0$, $\gamma_{1-}(-\infty) = +\infty$ and $\gamma_{1-}(0) = \frac{1}{\xi}$. Therefore, the equation $\gamma = \gamma_{1-}(x)$ has a negative root only for $\gamma > \frac{1}{\xi}$. □

Lemma 6. *For $\xi = 1$ all the eigenvalues of problem* (1)–(3) *in Case* 1 *with real γ are real. Each positive eigenvalue $\lambda_k(\gamma) = x_k^2(\gamma)$, where $x_k \in (p_{k-1}, p_k)$.*

Proof. The proof follows from Proposition 4 for the function γ_{1+}. □

Remark 15. *We enumerate the eigenvalues in such a way: $x_k(0) = \pi(k - \frac{1}{2})$, i.e., using the classical case.*

In this case, we get asymptotical properties of eigenvalues.

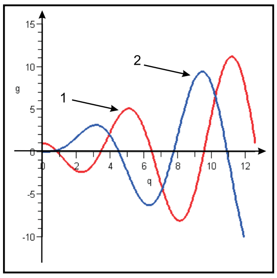
Corollary 2. *For problem* (1)–(3) *in Case* 1 *and* $\xi = 1$ *the properties*

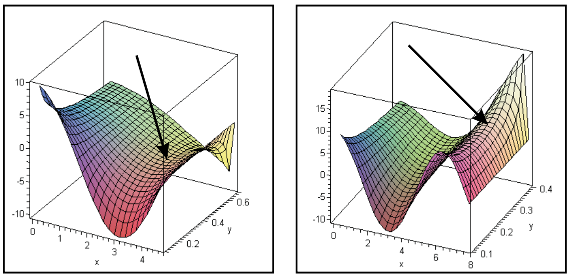
$$\lim_{\gamma\to-\infty} x_k(\gamma) = p_k, \quad \lim_{\gamma\to+\infty} x_k(\gamma) = p_{k-1}, k \in \mathbb{N}\setminus\{1\}, \quad \lim_{\gamma\to+\infty} x_1(\gamma) = -\infty$$

are valid.

In other cases ($\xi \in (0,1)$), the spectrum is not so simple. For real γ multiple and complex eigenvalues can exist. In many cases it is necessary to know when all eigenvalues are positive and non multiple, it means, when the analyzed problem spectrum is such as the classical problem. When the qualitative root distribution depends on the parameters γ and ξ, it is necessary to find such an interval for γ in which the spectrum of the problem satisfies this property.

The graphs of the functions $h_1(x) := \cos x - x\sin x$, $h_2(x) := \sin x - x\cos x$ for $x \geq 0$ are given in Fig. 4. Suppose that x_0, x_1, x_2 are the first three positive zeroes of the function h_1 and z_1 is the first positive zero of the function h_2. We define $\xi_k := \frac{\pi}{2x_k}$, $\gamma_k := x_k \cos x_k$ and $\tilde{\gamma} := \frac{z_1}{\sin z_1}$. Then $x_0 \approx 0.8603$, $x_1 \approx 3.4256$, $x_2 \approx 6.4373$, $\xi_1 \approx 0.4585$, $\xi_2 \approx 0.2440$, $\gamma_1 \approx -3.2884$, $\gamma_2 \approx 6.361$, $z_1 \approx 4.4934$, $\tilde{\gamma}_1 \approx -4.6033$.





(a) (b)

Fig. 4. Functions h_1 (graph. 1) and h_2 (graph. 2).

Fig. 5. Function $\gamma_{1+}(x,\xi)$.

Lemma 7. *If* $\gamma_1 \leq \gamma \leq \gamma_2$, *then all the eigenvalues of problem* (1)–(3) *are real for all* $\xi \in (0,1)$, *and limitary cases are realizable when* $\xi = \xi_2$ *and* $\xi = \xi_3$. *If* $\gamma_1 < \gamma \leq 1$, *then all the eigenvalues are positive and simple for all* $\xi \in (0,1)$.

Proof. We can consider only nonconstant eigenvalues, because the constant eigenvalues (if any) are positive. The function γ_{1+} defines the distribution of positive eigenvalues. We resolve γ_{1+} into multiplicands:

$$\gamma_{1+}(x;\xi) = \frac{x\cos x}{\sin(\xi x)} = g(x;\xi)\cos(x), \quad \text{where } g(x;\xi) := \frac{x}{\sin(\xi x)}.$$

The graphs of the functions $\gamma_{1+}(x;\xi)$, $\pm g(x;\xi)$ and $\pm x$ for various parameter ξ values are given in Fig. 6. As we can see, the graphs of the function $\gamma_{1+}(x;\xi)$ oscillate between the functions $g(x;\xi)$ and $-g(x;\xi)$. Since

$$g'(x) = \frac{\sin(\xi x) - \xi x\cos(\xi x)}{\sin^2(\xi x)} = \frac{h_2(\xi x)}{\sin^2(\xi x)}, \tag{26}$$

the minimum points of the function $|g(x)|$ are $x_{k,min} = \frac{z_k}{\xi}$, $k \in \mathbb{N}$, where z_k is the positive root of the equation $\sin z - z\cos z = 0$ and $g(x_{k,min}) = \frac{\tilde{\gamma}_k}{\xi}$.

We can find extremum points of the function $\gamma_{1+}(x,\xi)$ from a system

$$\begin{aligned}
\frac{\partial\gamma_{1+}}{\partial x} &= \frac{(\cos x - x\sin x)\sin(\xi x) - \xi x\cos x\cos(\xi x)}{\sin^2(\xi x)} = 0,\\
\frac{\partial\gamma_{1+}}{\partial \xi} &= -\frac{\xi x\cos x\cos(\xi x)}{\sin^2(\xi x)} = 0
\end{aligned}$$

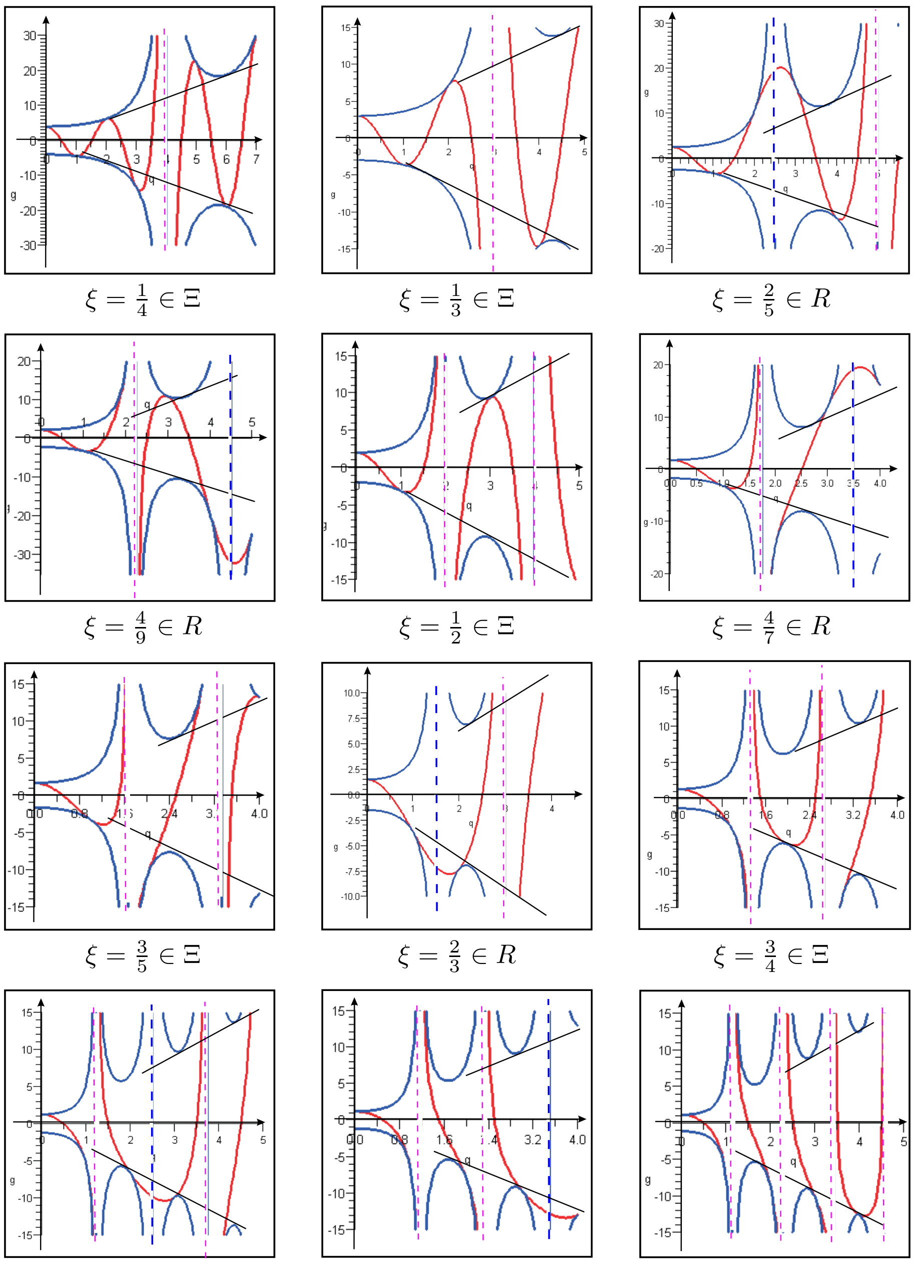
This system is equivalent to

$$\cos x - x\sin x = 0, \quad \cos(\xi x) = 0. \tag{27}$$

So, the extremum points are x_k, $k \in \mathbb{N}$ and don't depend on ξ $\big(x_0 \approx 0.8603$ does not satisfy the equation $\cos(\xi x) = 0$ for $\xi \in (0,1)\big)$. For x_1 we have $\xi_1 = \frac{\pi}{2x_1}$; for x_2 we have $\xi_2 = \frac{\pi}{2x_2}$ and $\xi_2' = \frac{3\pi}{2x_2} \approx 0.732$; for $x_3 \approx 9.5293$ there are three such $\xi_3 = \frac{\pi}{2x_3}$, $\xi_3' = \frac{3\pi}{2x_3}$, $\xi_3'' = \frac{5\pi}{2x_3}$, and so on.

Since $\gamma_{1+}'(c_k;\xi) = \frac{\gamma_{1+}(c_k;\xi)}{c_k} \neq 0$ (see Remark 9) has the same sign as the function, the constant eigenvalue points are not extremum points of the function $\gamma_{1+}(x,\xi)$ and they are not extremum points of the one-dimensional function $\gamma_{1+}(x;\xi)$ as well.

It follows from Lemma 4 that, for $|\gamma| \leq 1$, there are no complex γ-values of the function $\gamma = \frac{\cos x}{\sin(\xi x)}$. Consequently, there are no complex γ-values of the function $\gamma_{1+}(x;\xi)$ at the angle $|\gamma| \leq x$ for all $\xi \in (0,1)$, and we must prove this lemma only for $0 < x < \gamma_2$ when $\gamma > 0$, and $0 < x < |\gamma_1|$ when $\gamma < 0$. Since $|\gamma_1| < \gamma_2 < 3\pi \leq \frac{3\pi}{\xi}$, we investigate the function γ_{1+} for $x \in (0, \frac{3\pi}{\xi})$. The points $\tilde{x}_1 = \frac{\pi}{\xi}$ and $\tilde{x}_2 = \frac{2\pi}{\xi}$, $\tilde{x}_3 = \frac{3\pi}{\xi}$ can be poles or points of constant eigenvalues.



$\xi = \frac{1}{4} \in \Xi$ $\xi = \frac{1}{3} \in \Xi$ $\xi = \frac{2}{5} \in R$

$\xi = \frac{4}{9} \in R$ $\xi = \frac{1}{2} \in \Xi$ $\xi = \frac{4}{7} \in R$

$\xi = \frac{3}{5} \in \Xi$ $\xi = \frac{2}{3} \in R$ $\xi = \frac{3}{4} \in \Xi$

$\xi = \frac{4}{5} \in R$ $\xi = \frac{6}{7} \in R$ $\xi = \frac{7}{8} \in \Xi$

Fig. 6. Functions $\gamma_{1+}(x/\pi;\xi)$.

If $\xi > \frac{6}{7}$, then $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$ are poles and the function $\gamma_{1+}(x;\xi)$ is decreasing in each interval $(0,\tilde{x}_1)$, $(\tilde{x}_1,\tilde{x}_2)$, $(\tilde{x}_2,\tilde{x}_3)$. So, in this case all the γ-values are real. If $\xi \leq \frac{6}{7}$, then $\frac{2\pi}{\xi} \geq \frac{7\pi}{3} > \gamma_2$ and we investigate the function γ_{1+} for $x \in (0, \frac{2\pi}{\xi})$.

If $\frac{4}{5} < \xi \leq \frac{6}{7}$, then $\tilde{x}_1$, $\tilde{x}_2$ are poles and the function $\gamma_{1+}(x;\xi)$ is decreasing in each interval $(0,\tilde{x}_1)$, $(\tilde{x}_1,\tilde{x}_2)$. So, in this case, all the γ-values are real.

If $\xi = \frac{4}{5}$, then $\tilde{x}_1$ is a pole and $\tilde{x}_2 = c_1$ is a constant eigenvalue point. The function $\gamma_{1+}(x;\xi)$ is decreasing in each interval $(0,\tilde{x}_1)$, $(\tilde{x}_1,c_1)$ and $\gamma_{1+}(c_1) = -\frac{25\pi}{8} < \gamma_1$. Thus, in this case, all the γ-values are real for $\gamma_1 \leq \gamma \leq \gamma_2$.

If $\frac{2}{3} < \xi < \frac{4}{5}$ (see Fig. 6, $\xi = \frac{3}{4}$), then $\tilde{x}_1$, $\tilde{x}_2$ are poles. The function $\gamma_{1+}(x;\xi)$ is a decreasing function when $x \in (0,\tilde{x}_1)$ and has one (negative) local minimum point x_{min} when $x \in (\tilde{x}_1,\tilde{x}_2)$ and $\gamma_{1+}(x_{min};\xi) \leqslant g(z_1;\xi) = -\frac{\tilde{\gamma}_1}{\xi} \leqslant -\frac{5\tilde{\gamma}_1}{4} < \gamma_1$. So, in this case, the lemma is valid.

If $\xi = \frac{2}{3}$ (see Fig. 6, $\xi = \frac{2}{3}$) then $\tilde{x}_1 = q_1$ is a constant eigenvalue point and $\tilde{x}_2$ is a pole. The function $\gamma_{1+}(x;\xi)$ is decreasing when $x \in (0,c_1]$ and $\gamma_{1+}\big(c_1,\frac{2}{3}\big) = -\frac{9\pi}{4} < \gamma_1$ and has one local minimum point x_{min} when $x \in (c_1,\tilde{x}_2)$. So, in this case, the lemma is valid.

If $\xi < \frac{2}{3}$, then $|\gamma_1| < \frac{3\pi}{2} < \frac{\pi}{\xi}$ and for $\gamma < 0$ we have to prove that in the interval $(0, 3\pi/2)$ there are only real γ-values. In this interval, the function $\gamma_{1+}(x;\xi)$ has only one local minimum point x_{min}, and it is monotone in the intervals $(0,x_{min})$ and $(x_{min},3\pi/2)$ and $\gamma_{1+}(\pi/2;\xi) = \gamma_{1+}(3\pi/2;\xi) = 0$. For $x \in (\pi/2, 3\pi/2)$, we have only one extremum point (x_1,ξ_1) of the function $\gamma_{1+}(x,\xi)$ (see Fig. 5(a)) and $\gamma_{1+}(x_1,\xi_1) = \gamma_1$. This point is saddle point. Thus, we prove the lemma for negative γ. Note that the function $\gamma_{1+}(x;\xi)$ is a positive and monotone function for $x \in (0,\pi/2)$ and we consider this function for $x > \frac{3\pi}{2}$ and $\gamma > 0$.

If $\frac{4}{7} < \xi < \frac{2}{3}$, then $\tilde{x}_1$ and $\tilde{x}_2$ are poles. If $\xi = \frac{4}{7}$, then $\tilde{x_2} = c_1$ is a constant eigenvalue point. The function $\gamma_{1+}(x;\xi)$ increases for $x \in (3\pi/2,\pi/\xi)$ and $x \in (5\pi/2, 2\pi/\xi)$. If $\xi = \frac{4}{7}$, then $\gamma_{1+}(c_1,4/7) = \frac{49}{8}\pi > \gamma_2$.

If $\frac{4}{9} < \xi < \frac{4}{7}$, then $\tilde{x}_1$ and $\tilde{x}_2$ are poles. The function $\gamma_{1+}(x;\xi)$ increases for $x \in (3\pi/2,\pi/\xi)$ and has one local maximum point x_{max} for $x \in (5\pi/2, 7\pi/2)$ and $\gamma_{1+}(x_{max},\xi) \geq \frac{\tilde{\gamma}_1}{\xi} > \frac{9\tilde{\gamma}_1}{4} > \gamma_2$. If $\xi \leq \frac{4}{9}$, then $\frac{\pi}{\xi} > \frac{9\pi}{4} > \gamma_2$ and we can consider only $x \in (5\pi/2,\pi/\xi)$.

If $\frac{2}{5} < \xi < \frac{4}{9}$, then $\tilde{x}_1$ is a pole. If $\xi = \frac{4}{7}$, then $\tilde{x_2} = c_1$ is a constant eigenvalue point. The function $\gamma_{1+}(x;\xi)$ increases for $x \in (3\pi/2, \pi/\xi)$ and $x \in (5\pi/2, 2\pi/\xi)$. If $\xi = \frac{4}{7}$, then $\gamma_{1+}(c_1, 2/5) = \frac{25}{4}\pi > \gamma_2$.

If $\xi < \frac{2}{5}$, then $\gamma_2 < \frac{5\pi}{2} < \frac{\pi}{\xi}$ and for $\gamma > 0$ we have to prove that, in the interval $(3\pi/2, 5\pi/2)$ there are only real γ-values. In this interval the function $\gamma_{1+}(x;\xi)$ has only one local maximum point x_{max} and it is monotone in the intervals $(3\pi/2, x_{min})$ and $(x_{min}, 5\pi/2)$ and $\gamma_{1+}(3\pi/2;\xi) = \gamma_{1+}(5\pi/2;\xi) = 0$. For $x \in (3\pi/2, 5\pi/2)$, we have two function $\gamma_{1+}(x,\xi)$ extremum points (x_2, ξ_2) and (x_2, ξ_2'), but $\xi_2' > \frac{2}{5}$. We have a saddle point (see Fig. 5(a)) and $\gamma_{1+}(x_2, \xi_2) = \gamma_2$. Thus, we have proved the lemma for positive γ.

For $\gamma_1 \leq \gamma \leq \gamma_2$ the horizontal line γ intersects the graphs of the function γ_{1+}. If $\gamma = 0$, then we get the classical case with all positive and simple eigenvalues. When $\gamma_1 < \gamma < \gamma_2$, all eigenvalues remain real and simple. We can enumerate them just like in the classical case.

When $\gamma > \frac{1}{\xi}$ we have one negative eigenvalue. So, all eigenvalues will be positive for all $\xi \in (0,1)$ if $\gamma \leq 1$. □

Remark 16. *If $\xi = \xi_1$ and $\gamma = \gamma_1$ or $\xi = \xi_2$ and $\gamma = \gamma_2$, then we have one multiple eigenvalue.*

Remark 17. *In Fig. 7, we see how the function γ_1 transforms near the constant eigenvalue point for various ξ_k ($\xi_{k-1} < \xi_k$), $k = 1, 2, 3, 4, 5, 6$. In the case $k = 4$, we have a constant eigenvalue.*

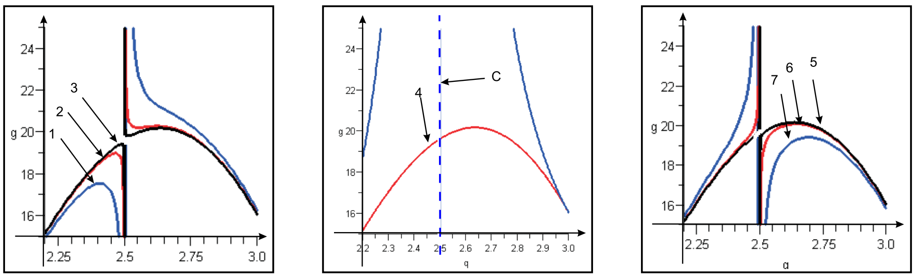


Fig. 7. Functions $\gamma_{1+}(x/\pi;\xi)$ near the constant eigenvalue point for various ξ_k, $k = 1, 2, 3, 4, 5, 6$.

Let $\tilde{p}_k = \frac{\pi}{\xi}k$, $k \in \mathbb{N}$, i.e., $\tilde{p}_k$ are poles or constant eigenvalue points. Then

$$\sigma_1(\gamma_{1+}, \tilde{p}_k) = (-1)^k \text{sign } \cos\left(\tfrac{1}{\xi}\pi k\right) = (-1)^k \text{sign } \cos \tilde{p}_k. \tag{28}$$

3.2 Real eigenvalues in Case 2

Proposition 5. *The function $\gamma_{2-}(x;\xi)$ is a monotone decreasing function for $x<0$ and all $\xi\in[0,1)$.*

Proof. For $\xi=0$, the function $\gamma_{2-}=\cosh x$ is a monotone decreasing function. The function $\gamma_{2-}(x)$ is even, when $x\in\mathbb{R}$, and $\gamma_{2-}(0)=1$ and $\gamma_{2-}(+\infty)=+\infty$. Therefore, we have to show that, in the interval $(0,+\infty)$, this function is increasing.

The function $y_1(x):=x\tanh x$, $x>0$ is a monotone increasing function as a product of two such functions.

Let's consider the function $y(\xi;x):=\xi\tanh(\xi x)-\tanh x$, $x>0$ and $\xi\in(0,1)$. For this function,

$$\lim_{\xi\to 0+} y(\xi;x)=-\tanh x<0,\qquad \lim_{\xi\to 1-} y(\xi;x)=0 \tag{29}$$

for all $x>0$. Its derivative with respect to ξ is equal to

$$y'(\xi;x)=\Big(\xi\tanh(\xi x)\Big)'=\frac{1}{x}\Big((\xi x)\tanh(\xi x)\Big)'>0.$$

Consequently, $y(\xi;x)$ is a monotone increasing function when $\xi\in(0,1)$, and from (29) we obtain that $y(\xi,x)<0$ for all $\xi\in(0,1)$ and all $x>0$.

The derivative of the function $\gamma_{2-}(x)$ is equal to

$$\frac{\sinh x\cosh(\xi x)-\xi\cosh x\sinh(\xi x)}{\cosh^2(\xi x)}=-\frac{\cosh x}{\cosh(\xi x)}\big(\xi\tanh(\xi x)-\tanh x\big)>0.$$

We see that, the function $\gamma_{2-}(x;\xi)$ is a monotone increasing function as $x>0$, and a monotone decreasing function when $x<0$. ☐

>From Proposition 5 we derive now the main result for a negative eigenvalue.

Lemma 8. *For $\gamma>\gamma_0=1$, there exists one negative eigenvalue, and for $\gamma\le\gamma_0$ there are no negative eigenvalues.*

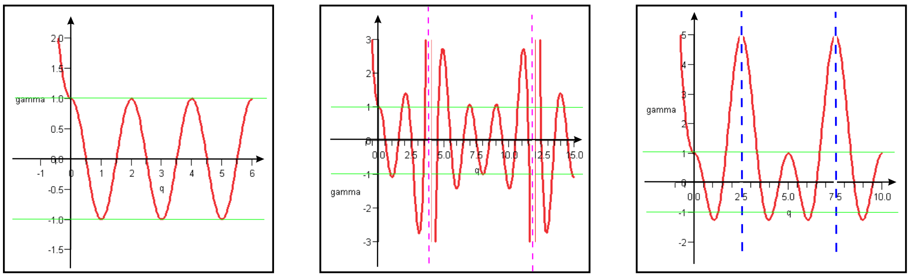
Proof. The function $\gamma_{2-}(x)$ is a monotone decreasing function when $x<0$, $\gamma_{2-}(-\infty)=+\infty$ and $\gamma_{2-}(0)=1$. Therefore, the equation $\gamma=\gamma_{2-}(x)$ has one negative root only for $\gamma>\gamma_0=1$ and there are no negative roots for $\gamma\le\gamma_0$. ☐

Another main result, in this case, is about real eigenvalues.

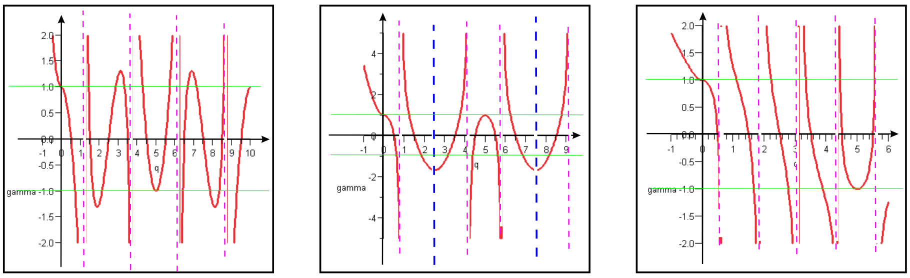
Lemma 9. *For* $|\gamma| \leq 1$*, all eigenvalues are real.*

Proof. The proof follows from Lemma 4 (Case 2). $\square$

If $|\gamma| \geq 1$, then there exist eigenvalues that can be multiple and complex. We can see some cases for various ξ in Fig. 8. In Fig. 9, we see how the function γ_2 transforms near the constant eigenvalue point.



$\xi = 0 \in \Xi$ $\qquad$ $\xi = \frac{1}{8} \in \Xi$ $\qquad$ $\xi = \frac{1}{5} \in R$



$\xi = \frac{2}{5} \in \Xi$ $\qquad$ $\xi = \frac{3}{5} \in R$ $\qquad$ $\xi = \frac{4}{5} \in \Xi$

Fig. 8. Functions $\gamma_2(x/\pi; \xi)$.

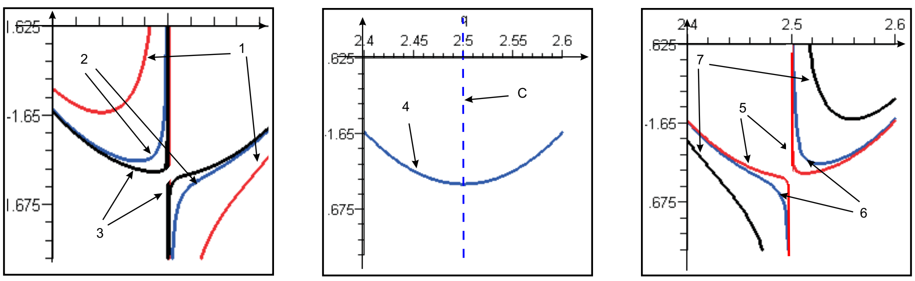


Fig. 9. Functions $\gamma_{2+}(x/\pi; \xi)$ near the constant eigenvalue point for various ξ_k, $k = 1, 2, 3, 4, 5, 6$.

Let $\tilde{p}_k = \frac{\pi}{\xi}(k - \frac{1}{2})$, $k \in \mathbb{N}$, i.e., $\tilde{p}_k$ are poles or constant eigenvalue points, then

$$\sigma_1(\gamma_{2+}, \tilde{p}_k) = (-1)^{k-1}\text{sign}\,\cos\left(\tfrac{1}{\xi}\pi(k - \tfrac{1}{2})\right) = (-1)^{k-1}\text{sign}\,\cos\tilde{p}_k. \quad (30)$$

3.3 Real eigenvalues in Case 3

The real spectrum in Case 3 is more complicated (see, Fig. 10). In this case $\gamma_0 = 1$. When γ is real, multiple and complex eigenvalues can exist for all $\gamma \neq 0$. For example, if $\xi = \frac{1}{2}$, then $\gamma_{3+}(x) = \frac{2}{x}\sin(x/2)$ and $|\gamma_{3+}| \leq \frac{2}{x}$.

Proposition 6. *The function $\gamma_{3+}(x;1)$ is a monotone increasing function in each interval (p_{k-1}, p_k); the function $\gamma_{3-}(x;1)$ is a monotone increasing function for $x < 0$. The function $\gamma_{3-}(x;\xi)$ is a monotone decreasing function for $x < 0$ only for $\xi \in [0, \sqrt{3}/3]$ and has one local minimum point $x_{\min} \in (-\infty, 0)$ for $\xi \in (\sqrt{3}/3, 1)$.*

Proof. The functions $\gamma_{3-}(x;1) = 1/\gamma_{1-}(x;1)$, $\gamma_{3+}(x;1) = 1/\gamma_{1+}(x;1)$. Thus, we get the proof for $\xi = 1$ from Proposition 4.

In Proposition 4 we show (see, (25)) that $\frac{\sinh x}{\sinh(\xi x)}$ is an increasing positive function for all $x > 0$ and $\xi \in (0,1)$. So, the function

$$y_1(x;\xi_1,\xi_2) := \frac{\sinh(\xi_1 x)}{\sinh(\xi_2 x)}, \quad x > 0, \quad 0 < \xi_2 < \xi_1,$$

is increasing and positive too. Since $\lim_{x\to 0} y_1(x;\xi_1,\xi_2) = \xi_1/\xi_2 > 0$, we have

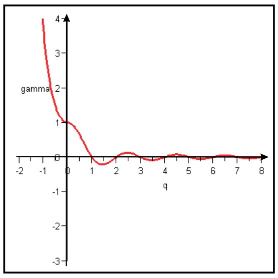
$$\xi_2\sinh(\xi_1 x) - \xi_1\sinh(\xi_2 x) > 0, \quad \text{for } x > 0, \quad 0 < \xi_2 < \xi_1. \quad (31)$$

Let us consider the positive function

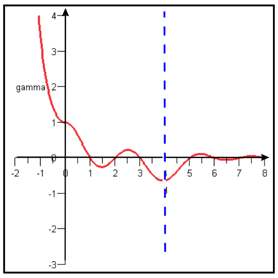
$$y_2(x;\xi_1,\xi_2) := \frac{\tanh(\xi_2 x)}{\tanh(\xi_1 x)}, \quad x > 0, \quad 0 < \xi_2 < \xi_1.$$

Its derivative with respect to x is

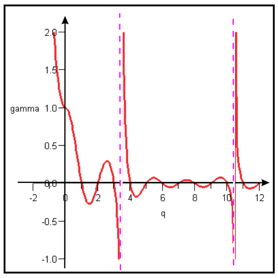
$$y_2'(x;\xi_1,\xi_2) = \frac{\xi_2\sinh(2\xi_1 x) - \xi_1\sinh(2\xi_2 x)}{2\sinh^2(\xi_1 x)\cosh^2(\xi_2 x)} > 0, \quad \text{for } x > 0, \quad 0 < \xi_2 < \xi_1.$$



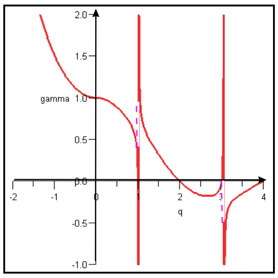
$\xi = 0 \in \Xi$



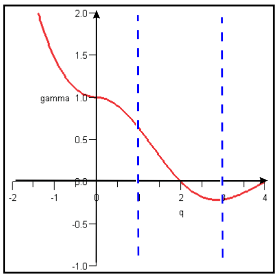
$\xi = \frac{1}{8} \in R$



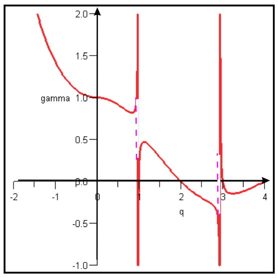
$\xi = \frac{1}{7} \in \Xi$



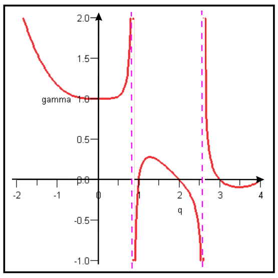
$\xi = 0.49 \in R$



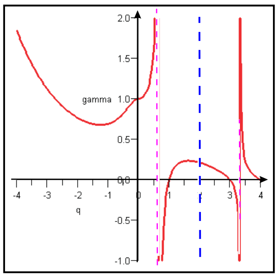
$\xi = \frac{1}{2} \in R$



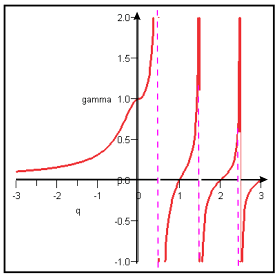
$\xi = 0.51 \in R$



$\xi = \frac{\sqrt{3}}{3} \in \Xi$



$\xi = \frac{3}{4} \in R$



$\xi = 1 \in \Xi$

Fig. 10. Functions $\gamma_3(x/\pi; \xi)$.

Hence, we get that

$$y_3(x; \xi) := \frac{\tanh(\frac{1}{2}x)}{\xi \tanh(\xi x)}, \quad x > 0,$$

is an increasing positive function for all $\xi \in (\frac{1}{2}, 1)$.

For the function

$$y_4(x) := 2 \sinh x \cosh x + \sinh x - 3x \cosh x,$$

we have $y_4(0) = 0$, $y_4'(0) = 0$ and for $x > 0$

$$\begin{aligned} y_4''(x) &= 8\sinh x \cosh x - 5\sinh x - 3x\cosh x \\ &= 3\cosh x(\sinh x - x) + 5\sinh x(\cosh x - 1) > 0. \end{aligned}$$

As a result, the function $y_4(x)$, $x > 0$ is positive . For the function

$$y_5(x) := \sinh x \cosh x - \sinh x - x^2 \sinh x + x\cosh x - x,$$

we have $y_5(0) = 0$, $y_5'(0) = 0$ and for $x > 0$

$$\begin{aligned} y_5''(x) &= 4\sinh x \cosh x - \sinh x - 3x\cosh x - x^2 \sinh x \\ &= 2\sinh(\cosh x - 1 - \tfrac{1}{2}x^2) + 2\sinh x\cosh x + \sinh x - 3x\cosh x \\ &= 2\sinh(\cosh x - 1 - \tfrac{1}{2}x^2) + y_4(x) > 0. \end{aligned}$$

Consequently, the function $y_5(x)$, $x > 0$ is positive, too.

Note that the function $x\coth x - 1 = (x\cosh x - \sinh x)/\sinh x > 0$. Then the derivative of the positive function

$$y_6(x) := \frac{x\coth x - 1}{x\tanh(\frac{1}{2}x)} = \frac{x\cosh x - \sinh x}{x(\cosh x - 1)}, \quad x > 0,$$

is equal to

$$y_6'(x) = \frac{\sinh x(\cosh x - 1 - x^2) + x(\cosh x - 1)}{x^2(\cosh x - 1)^2} = \frac{y_5(x)}{x^2(\cosh x - 1)^2} > 0.$$

So, the function

$$y(x;\xi) := \frac{x\coth x - 1}{x\xi\tanh(\xi x)} = y_6(x) \cdot y_3(x;\xi)$$

is a positive increasing function for all $x \geq 0$, $\xi \in (\frac{1}{2}, 1)$ and

$$y_0 := \lim_{x\to 0} y(x;\xi) = \frac{1}{3\xi^2}; \quad y_\infty := \lim_{x\to+\infty} y(x;\xi) = \frac{1}{\xi} > 1.$$

If $\xi \in (\frac{1}{2}, \sqrt{3}/3]$, then $y_0 \geqslant 1$ and $y(x;\xi) > 1$ for all $x > 0$; if $\xi \in (\sqrt{3}/3, 1)$, then $y_0 < 1$ and there exists $x_{\min} = x_{\min}(\xi) > 0$ such that $y(x_{\min};\xi) = 1$ and $y(x;\xi) < 1$ for all $0 < x < x_{\min}$, $y(x;\xi) > 1$ for all $x > x_{\min}$.

We reformulate these properties for the function

$$f(x;\xi) := x\coth x - 1 - x\xi\tanh(\xi x),$$

i.e., if $\xi \in (\frac{1}{2}, \sqrt{3}/3]$, then $f(x;\xi) > 0$ for all $x > 0$; if $\xi \in (\sqrt{3}/3, 1)$, then there exists $x_{\min} = x_{\min}(\xi) > 0$ such that $f(x_{\min};\xi) = 0$ and $f(x;\xi) < 0$ for all $0 < x < x_{\min}$, $f(x;\xi) > 0$ for all $x > x_{\min}$.

Since

$$\frac{\partial}{\partial\xi}\bigl(x\xi\tanh(\xi x)\bigr) = x\tanh(\xi x) + \frac{x\xi^2}{\cosh^2(\xi x)} > 0, \quad \text{for } x > 0,$$

we obtain $\xi x\tanh(\xi x) < \frac{1}{2}x\tanh(\frac{1}{2}x)$, and, for $\xi \in [0, \frac{1}{2}]$, we estimate

$$f(x;\xi) \geqslant x\coth x - 1 - \tfrac{1}{2}x\tanh(\tfrac{1}{2}x) = \tfrac{1}{2}x\coth(\tfrac{1}{2}x) - 1 > 0.$$

Finally, we have

$$\gamma'_{3-}(x;\xi) = \frac{x\coth x - 1 - x\xi\tanh(\xi x)}{x^2\cosh^2(\xi x)} = f(x;\xi)\frac{\sinh x}{x^2\cosh(\xi x)}.$$

The function $\gamma_{3-}(x;\xi)$, $x \in \mathbb{R}$ is an even function. Therefore, monotonicity properties of the function $\gamma_{3-}(x;\xi)$, $x < 0$, follow from the properties of the function $f(x;\xi)$: if $\xi \in [0, \sqrt{3}/3]$, then $\gamma_{3-}(x;\xi)$ is a decreasing function for $x \leqslant 0$; if $\xi \in (\sqrt{3}/3, 1)$, then there exists $x_{\min} = x_{\min}(\xi) < 0$ such that $\gamma_{3-}(x;\xi)$ is a decreasing function for and $x \leqslant x_{\min}$, and $\gamma_{3-}(x;\xi)$ is an increasing function for $x_{\min} \leqslant x \leqslant 0$; if $\xi = 1$, then $\gamma_{3-}(x;\xi)$ is an increasing function for $x \leqslant 0$. ☐

Lemma 10. *If $\xi \in [0, \sqrt{3}/3]$, then there exists one negative eigenvalue only for $\gamma > \gamma_0$. If $\xi \in (\sqrt{3}/3, 1)$, then there exists $x_{\min} < 0$ and $\gamma_* = \gamma_{3-}(x_{\min};\xi) \in (0, \gamma_0)$ such that there exists one double negative eigenvalue for $\gamma = \gamma_*$ and only one simple eigenvalue for $\gamma > \gamma_0$, two negative eigenvalues exist for $\gamma \in (\gamma_*, \gamma_0)$, and for $\gamma < \gamma_*$, there are no negative eigenvalues. If $\xi = 1$, then there exists one negative eigenvalue only for positive $\gamma < \gamma_0$, but there are no negative eigenvalues for $\gamma \geqslant \gamma_0$.*

Proof. The function $\gamma_{3-}(x;\xi)$ is positive. From Proposition 6 and the conditions

$$\gamma_{3-}(-\infty) = \begin{cases} +\infty, & \text{for } \xi < 1, \\ 0, & \text{for } \xi = 1; \end{cases} \qquad \gamma_{3-}(0) = 1,$$

we get the proof of this lemma. ☐

Remark 18. *In Fig.* 11*, we see how the function* γ_3 *transforms near the constant eigenvalue point for various* ξ_k*,* $k = 1, 2, 3, 4, 5, 6$*. In the case* $k = 4$*, we have a constant eigenvalue.*

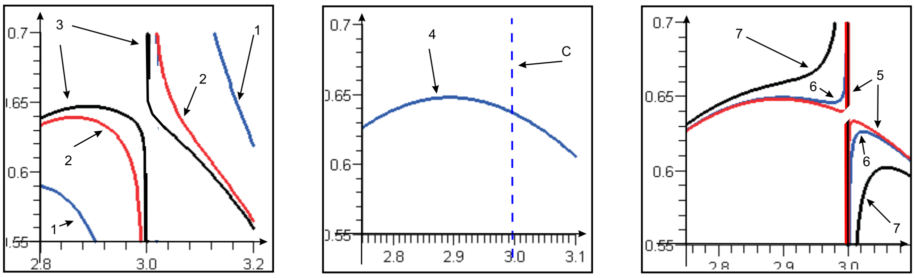


Fig. 11. Functions $\gamma_{3+}(x/\pi;\xi)$ near the constant eigenvalue point for various ξ_k, $k = 1, 2, 3, 4, 5, 6$.

Let $\tilde{p}_k = \frac{\pi}{\xi}(k - \frac{1}{2})$, $k \in \mathbb{N}$, i.e., $\tilde{p}_k$ be poles or constant eigenvalue points. Then

$$\sigma_1(\gamma_{1+}, \tilde{p}_k) = \begin{cases} (-1)^k \text{sign} \sin \tilde{p}_k, & \text{for } \xi > 0, \\ 0, & \text{for } \xi = 0. \end{cases} \tag{32}$$

4 Conclusions

- Sturm-Liouville problems (1)–(3) (Cases 1–3) have similar spectrum properties in the complex plane. Spectrums of these problems have no constant eigenvalues for irrational ξ and for some rational $\xi \in \Xi$ and have a countable number of nonconstant and constant eigenvalues for rational $\xi \in R$. All constant eigenvalues are real positive numbers.

- In Cases 1 and 2, the problems have only one negative eigenvalue for $\gamma > \gamma_0$. In Case 3, there exists one negative eigenvalue only for $\xi \leq \sqrt{3}/3$ and $\gamma > 1$, and for $\xi = 1$ and $0 < \gamma < 1$. In Case 3, we have two negative eigenvalues for $\xi \in (\sqrt{3}/3, 1)$ and $0 < \gamma_* < \gamma < \gamma_0 = 1$.

- Positive parts of the spectrums are different for the real γ case. For the problems in Cases 1 and 2, all real eigenvalues exist only for $\gamma_m(\xi) \leq \gamma \leq \gamma_M(\xi)$, but the interval $[\bar{\gamma}_m, \bar{\gamma}_M] \subset [\gamma_m, \gamma_M]$ is the same for all ξ. In Case 3, for every $\gamma \neq 0$ and $\xi < 1$, multiple and complex eigenvalues can exist and, only for $\xi = 1$, all eigenvalues are real.

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